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Sigma-model approach in General Relativity and its generalizations

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SIGMA-MODEL APPROACH IN GENERAL RELATIVITY AND ITS GENERALIZATIONS

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Construction of exact solutions of Einstein equations in General Relativity and its generalizations, such as supergravities and string inspired models in various dimensions is an important problem of theoretical and mathematical physics. Here we review one of the most efficient tools which is applicable to large class of theories involving metric, scalar, vector and antisymmetric form fields without higher-derivative terms, scalar potentials and cosmological constant. This includes pure gravity in arbitrary dimensions, bosonic parts of supergravities, as well as many non-supersymmetric gravity-matter systems. The method is based on dimensional reduction of D-dimensional Einstein equations assuming the solutions to depend only on three independent variables in a way that the field equations reduce to those of the three-dimensional gravity coupled scalar sigma-model. Typically this amounts to starting with the Einstein-Hilbert action coupled to scalar-vector/antisymmetric form sector with Maxwell-dilaton type action and restricting to field configurations admitting $D-3$ commuting Killing vectors. The corresponding dimensional reduction is commonly referred as toroidal.

Keywords: general relativity, sigma-models, supergravity, Einstein equations.

1. Stationary D=4 vacuum Einstein equations

Consider four-dimensional spacetime admitting (at least) one Killing vector field. Such spacetime is called stationary if this field is timelike in an essential domain. The following is equally applicable if the Killing vector is spacelike (this is suggested as an exercise), but not null. The interval in appropriate coordinates (called Kaluza-Klein) can be presented as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(dx^0 + a_i dx^i)^2 + f^{-1} h_{ij} dx^i dx^j\,, \tag{1}$$

where the metric function $g_{00} = f$, the three-dimensional reduced metric h_{ij} ($\mu = 0, i;\ i = 1,2,3$) and the Kaluza-Klein (KK) three-dimensional (co)vector a_i depend only on the x^i. One can introduce a 3– vector τ^i, dual to the rotation 2–form $da = \partial_i a_j dx^i \wedge dx^j$,

$$\tau^i = -f^2 \frac{\epsilon^{ijk}}{\sqrt{h}} \partial_j a_k, \tag{2}$$

assuming further that the indices of all three–dimensional quantities are raised and lowered with the metric h_{ij} while for four–dimensional tensors one still uses $g_{\mu\nu}$. Then the components of the four-dimensional Ricci tensor can be expanded as

$$R_{00} = \frac{1}{2}\left(f\Delta f - (\nabla f)^2 + \tau^2\right), \quad R_0^i = \frac{f}{2\sqrt{h}} \epsilon^{ijk} \partial_j \tau_k\,, \tag{3}$$

$$R_{ij} = f^2 \mathscr{R}_{ij} - \frac{1}{2}\left[(\partial_i f)(\partial_j f) + \tau_i \tau_j - h_{ij}\left(f\Delta f - (\nabla f)^2 + \tau^2\right)\right]\,, \tag{4}$$

where $\mathscr{R}_{ij}(h)$ is the Ricci tensor of the theree–space, ∇_i denotes 3–covariant derivative, $\Delta = \nabla^2$, and 3–vector scalar products are understood with respect to the metric h_{ij}. The mixed component Einstein equation $R_0^i = 0$ can be solved by introducing the twist (pseudo)scalar potential χ:

$$\tau_i = \partial_i \chi \,. \tag{5}$$

Then the $R_{00} = 0$ equation reduces to

$$f\Delta f - (\nabla f)^2 + (\nabla\chi)^2 = 0\,, \tag{6}$$

taking $\nabla_i \tau_i$ and using this one get similar equation for χ:

$$f\Delta\chi - 2\nabla f \nabla\chi = 0\,, \tag{7}$$

and the remaining equation $R_{ij} = 0$ to three-dimensional Einstein equation

$$\mathscr{R}_{ij}(h) = \frac{1}{2f^2}\Big(\nabla_i f \nabla_j f + \nabla_i \chi \nabla_j \chi\Big). \tag{8}$$

Obviously $\nabla_i = \partial_i$ when applied to scalars.

2. Sigma-model representation

Eqs. (6), (7) and (8) can be equally derived from the three-dimensional action

$$S_\sigma = \int \Big(-\mathscr{R}(h) + G_{AB}(X)\nabla_i X^A \nabla_j X^B h^{ij}\Big)\sqrt{h}d^3x\,, \tag{9}$$

which defines the three-dimensional gravitating sigma model based on the two-dimensional target space parameterized by coordinates $X^A,\ A = 1,2,\ X^1 = f, X^2 = \chi$ and endowed with the metric G_{AB}, the corresponding line element being

$$dl_T^2 \equiv G_{AB}dX^A dX^B = \frac{1}{2f^2}(df^2 + d\chi^2)\,. \tag{10}$$

Thus the g_{00} component of the four-dimensional metric and the twist potential obtained by dualization of the KK vector act as matter sources for the three-dimensional metric h_{ij}. Generally the "matter" equations reads:

$$\nabla_j\left(G_{AB}h^{ij}\nabla_i X^A\right) = \frac{1}{\sqrt{h}}\partial_j\left(G_{AB}h^{ij}\sqrt{h}\partial_i X^A\right) = 0\,, \tag{11}$$

This defines the harmonic map between the coordinate three space with metric h_{ij} and the target space with metric G_{AB}.

Let us show that the target space (10) is a symmetric riemmanian space. Consider the Killing equations

$$K_{(A;B)} = 0\,, \tag{12}$$

where the semi-colon denotes the covariant derivative with respect to G_{AB}. For the metric (10) one finds three Killing vectors

$$H = 2(f\partial_f + \chi\partial_\omega)\,, \quad E_+ = \partial_\chi\,, \quad E_- = (\chi^2 + f^2)\partial_\chi + 2\chi f\partial_f\,, \tag{13}$$

These three Killing vectors generate the Lie algebra $sl(2,R) = so(1,2)$:

$$[H, E_\pm] = \pm 2E_\pm\,, \quad [E_+, E_-] = H\,, \tag{14}$$

revealing that the target space is the symmetric coset space $SL(2,R)/SO(2)$ (this identification follows from consideration of its signature). The first Killing vector M describes scale transformation $f,\chi \to a^2(f,\chi)$, the second — gauge transformation $\chi \to \chi + b$, obvious from the dualization equation (Rfdual), the third corresponds to non-trivial Ehlers transformation :

$$(\chi - if)^{-1} \to (\chi - if)^{-1} + c\,, \tag{15}$$

where a, b, c are real parameters.

3. Matrix representation of the coset

In this example obtaining finite isometries of the target space is rather simple, but for larger cosets it is better to use matrix representation. For generators one can use Pauli matrices

$$H = \sigma_3, \quad E_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)\,. \tag{16}$$

Tho obtain matrix representative of the coset one can exponentiate the Borel subalgebra spanned by upper triangle matrices H, E_+ (including diagonal) with $\xi = \ln f, \chi$ as coefficients. Exponentiation of these leads again to some upper triangle matrix:

$$v = e^{\frac{1}{2}\xi H} e^{\chi E_+} = \begin{pmatrix} e^{\frac{1}{2}\xi} & \chi e^{\frac{1}{2}\xi\omega} \\ 0 & e^{-\frac{1}{2}\xi} \end{pmatrix}. \tag{17}$$

Since infinitesimally

$$dv v^{-1} = \frac{1}{2} d\xi H + e^{\xi} d\chi E_+\,, \tag{18}$$

one can express the target space metric using square of this element. But the upper triangle gauge is not preserved by isometry transformation, so it is better to chose as the representative of the coset a gauge invariant matrix

$\mathcal{M} = v^T v$. Coming back to f, we obtain

$$\mathcal{M} = \begin{pmatrix} f + f^{-1}\chi^2 & -f^{-1}\chi \\ -f^{-1}\chi & f^{-1} \end{pmatrix}, \tag{19}$$

so the target space metric can be rewritten as

$$dl_T^2 = \frac{1}{4}\mathrm{Tr}(\mathcal{M}^{-1} d\mathcal{M} \mathcal{M}^{-1} d\mathcal{M})\,, \tag{20}$$

and the vacuum four-dimensional Einstein equations read

$$\nabla(\mathcal{M}^{-1}\nabla\mathcal{M}) = 0\,, \tag{21}$$

$$\mathcal{R}(h) = \tfrac{1}{4}\mathrm{Tr}(\mathcal{M}^{-1}\partial_i\mathcal{M}\mathcal{M}^{-1}\partial_j\mathcal{M})\,. \tag{22}$$

Denoting more generally the isometry group as G and its isotropy subgroup as H (not to confuse with Cartan subalgebra generator) we see, that the equations remain invariant under transformation of the coset matrix $\mathcal{M}(x) \in G//H$

$$\mathcal{M}(x) \to \mathcal{M}'(x) = U^T \mathcal{M}(x) U\,, \quad (U \in G)\,, \tag{23}$$

so this symmetry can be used to generate new solution form some known one. Choosing $U \in H$ one preserve some point in the target space, this may be used to generate one asymptotically flat (AF) solution from another AF one.

4. Ernst equations

The target space (10) is Kähler. To see this it is enough to introduce the complex coordinate, the (vacuum) Ernst potential,

$$\epsilon = i f - \chi, \tag{24}$$

(the original Ernst potential differs from this by i). The Kähler metric $G_{\epsilon\bar{\epsilon}}$ is generated by the Kähler potential K

$$G_{\epsilon\bar{\epsilon}} = \partial_\epsilon \partial_{\bar{\epsilon}} K(\epsilon, \bar{\epsilon}), \tag{25}$$

for which one obtains the following simple expression

$$K = -\ln V, \quad V = \mathrm{Im}\epsilon = f. \tag{26}$$

Thus the Kähler potential for any stationary solution of the vacuum Einstein equation is directly related to the g_{00} component of the four–dimensional metric. The harmonic map now is effected via Ernst equation

$$(\epsilon - \bar{\epsilon})\Delta\epsilon = 2(\nabla\epsilon)^2. \tag{27}$$

The $SL(2,R)$ isometry of the target space is manifest:

$$\epsilon \to \frac{a\epsilon + b}{c\epsilon + d}, \quad ad - bc = 1, \tag{28}$$

with real a, b, c, d, while its one–parameter subgroups are

$$i) \qquad \text{gauge}: \quad \epsilon \to \epsilon + \mathrm{b}, \quad (\mathrm{a} = \mathrm{d} = 1,\ \mathrm{c} = 0), \tag{29}$$

$$ii) \qquad \text{scale}: \quad \epsilon \to \mathrm{a}^2\epsilon, \quad (\mathrm{b} = \mathrm{c} = 0,\ \mathrm{d} = 1/\mathrm{a}), \tag{30}$$

preserving the r.h.s. of the three–dimensional Einstein equations , but modifies the four–dimensional metric, thus producing physically inequivalent field configurations (in particular, transforming AF solutions into non –AF);

$$iii) \qquad \text{Ehlers}: \quad \epsilon^{-1} \to \epsilon^{-1} + \mathrm{c}, \quad (\mathrm{d} = \mathrm{a} = 1,\ \mathrm{b} = 0). \tag{31}$$

This transformation an essential part of the whole group. It is non–linear being expressed in terms of ϵ, and physically corresponds to mixing of a mass and a NUT charge (gravitational analog of electric–magnetic duality) [3].

5. Problems

1. Generate Taub-NUT metric and express it in Weyl-Papapetrou form

$$\begin{aligned}
ds^2 &= e^{2U}(dt + Ad\varphi)^2 - e^{-2U}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right], \qquad (32)\\
e^{2U} &= \frac{(R_+ + R_-)^2 + 4(m^2 + n^2)}{(R_+ + R_- + 2m)^2 + 4n^2}, \quad e^{2\gamma} = \frac{(R_+ + R_-)^2 + 4(m^2 + n^2)}{4R_+ R_-},\\
A &= -\frac{n}{\sqrt{m^2 + n^2}}(R_+ - R_-), \quad R_\pm^2 = \rho^2 + \left(z \pm \sqrt{m^2 + n^2}\right)^2,
\end{aligned}$$

applying Ehlers transformation to Schwarzschild metric with mass m.

2. Show that γ in the Weyl-Papapetrou metric satisfies

$$\begin{aligned} 4\gamma_\rho &= \rho f^{-2}\left(\epsilon_\rho\bar{\epsilon}_\rho - \epsilon_z\bar{\epsilon}_z\right), \\ 4\gamma_z &= \rho f^{-2}\left(\epsilon_\rho\bar{\epsilon}_z + \epsilon_z\bar{\epsilon}_\rho\right). \end{aligned} \tag{33}$$

3. Express the metric (32) in prolate spheroidal coordinates $\rho^2 = l^2(x^2-1)(1-y^2)$, $z = lxy$, show that the potential

$$\epsilon = \frac{ipx - qy - i}{ipx + qy + i}$$

satisfies Ernst equation and derive the corresponding 4-metric (Kerr)

6. Einstein-Maxwell: $S = \int\left(R - F^2\right)\sqrt{-g}\, d^4x$

Maxwell equations and Bianchi identities

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0, \quad \partial_\nu(\sqrt{-g}\tilde{F}^{\mu\nu}) = 0, \tag{34}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}E^{\mu\nu\lambda\tau}F_{\lambda\tau}$, $E^{\mu\nu\lambda\tau} = \epsilon^{\mu\nu\lambda\tau}\sqrt{-g}$ with assumption of stationarity are partially solved introducing the electric v and magnetic u potentials

$$F_{i0} = \frac{1}{\sqrt{2}}\partial_i v, \quad F^{ij} = \frac{f}{\sqrt{2h}}\epsilon^{ijk}\partial_k u. \tag{35}$$

The remaining components of the $F^{\mu\nu}$ can be expressed in terms of v and u using the relation

$$F^{i0} = F^{ij}\omega_j - h^{ij}F_{j0}, \tag{36}$$

where h^{ij} is the 3-inverse of h_{ij}. Another useful relation is

$$F_{ij} = f^{-2}h_{ik}h_{jl}F^{kl} + 2F_{0[i}\omega_{j]}. \tag{37}$$

From the mixed component of Einstein equation now with the source term $8\pi T_0^i = \frac{f}{\sqrt{h}}\epsilon^{ijk}(\nabla_j v)(\nabla_k a)$, one finds

$$\tau_i = w_i + \nabla_i\chi, \quad w_i = v\nabla_i u - u\nabla_i v, \tag{38}$$

Proceeding as before we derive three-dimensional sigma-model with the target space metric parameterized by $X^A = f, \chi, v, u$:

$$dl_T^2 = G_{AB}dX^A dX^B = \frac{1}{2f^2}\left(df^2 + (d\chi + vdu - udv)^2\right) - \frac{1}{f}(dv^2 + du^2). \tag{39}$$

Problem: Show that Killing equations for G_{AB} are solved by eight Killing vectors, forming the algebra $su(1,2)$

$$
\begin{aligned}
K_1 &= v\partial_u - u\partial_v, \quad K_2 = 2(f\partial_f + \chi\partial\chi) + u\partial_u + v\partial_v \\
K_3 &= \partial_u + v\partial_\chi, \quad K_4 = \partial_v - u\partial_\chi, \quad K_5 = 2\partial_\chi, \\
K_6 &= 2fv\partial_f + \left(v\chi + uF\right)\partial_\chi + \left(\frac{1}{2}(v^2 - 3u^2) + f\right)\partial_v + \chi_{-2}\partial_u \\
K_7 &= 2fu\partial_f + \left(u\chi - vF\right)\partial_f + \left(\frac{1}{2}(u^2 - 3v^2)) + f\right)\partial_u - \chi_2\partial_v, \\
K_8 &= 2f\chi\partial_f + (\chi^2 - F^2)\partial_\chi + (v\chi - uF)\partial_v + (u\chi + vF)\partial_u,
\end{aligned}
\tag{40}
$$

where $F = f - (v^2 + u^2)/2$ and $\chi_n \equiv \chi - nuv$. The last three correspond to Harrison charging transformations and Ehlers transformation.

Introduce the EM Ernst potentials (in original version now)

$$
\mathscr{E} = f + i\chi - \overline{\Phi}\Phi, \qquad \Phi = \frac{1}{\sqrt{2}}(v + iu), \tag{41}
$$

in terms of which the target space metric reads

$$
dl^2 = \frac{1}{2f^2}|\,d\mathscr{E} + 2\overline{\Phi}\,d\Phi|^2 - \frac{2}{f}\,d\Phi\,d\overline{\Phi}. \tag{42}
$$

This is the metric of the symmetric space SU(2,1)/S(U(2) × U(1)). A matrix representative of this coset may be chosen as

$$
M = f^{-1}\begin{pmatrix} 1 & \sqrt{2}\Phi & i(\overline{\mathscr{E}} - \mathscr{E} + 2\Phi\overline{\Phi})/2 \\ \sqrt{2}\overline{\Phi} & -(\mathscr{E} + \overline{\mathscr{E}} - 2\Phi\overline{\Phi})/2 & -i\sqrt{2}\mathscr{E}\overline{\Phi} \\ i(\overline{\mathscr{E}} - \mathscr{E} - 2\Phi\overline{\Phi})/2 & i\sqrt{2}\overline{\mathscr{E}}\Phi & \mathscr{E}\overline{\mathscr{E}}. \end{pmatrix} \tag{43}
$$

Problems: 1. Derive finite transformations corresponding to Harrison and Ehlers infinitesimal ones as given by K_6, K_7, K_8

2. Derive Kerr-Newman solution from Kerr solution using SU(2,1) transformations

7. Einstein-Maxwell-Dilaton-Axion

Consider a four–dimensional action

$$
S = \int \left\{ -R + 2\left|\partial z(z - \bar{z})^{-1}\right|^2 + \left(iz\mathscr{F}^n_{\mu\nu}\mathscr{F}^{n\mu\nu} + c.c \right) \right\} \sqrt{-g}\,d^4x, \tag{44}
$$

where $\mathscr{F}^n = (F^n + i\tilde{F}^n)/2$, $\tilde{F}^{n\mu\nu} = \frac{1}{2}E^{\mu\nu\lambda\tau}F^n_{\lambda\tau}$, $n = 1, ..., p$, and the sum over repeated n is understood. For $p = 6$ this is the bosonic sector of $N = 4, D = 4$ supergravity. This action is invariant under $SO(p)$ rotations of vector fields, which is an analog of T–duality of dimensionally reduced theories. The equations of motion and Bianchi identities (but not the action) are also invariant under S–duality transformations

$$
\begin{gathered}
z \to \frac{az + b}{cz + d}, \quad ad - bc = 1, \\
F^n \to (c\kappa + d)F^n + c\mathrm{e}^{-2\phi}\tilde{F}^n.
\end{gathered}
\tag{45}
$$

Introduce electric v^n and magnetic u^n scalar potentials for F_n as follows

$$F^n_{i0} = \frac{1}{\sqrt{2}}\partial_i v^n, \quad 2\mathrm{Im}\left(z\mathscr{F}^{nij}\right) = \frac{f}{\sqrt{2h}}\epsilon^{ijk}\partial_k u^n. \tag{46}$$

In three dimensions the rotation one form ω_i plays a role of the graviphoton, and one can show using the standard argument that the "T–duality" group is enlarged to $SO(1,p+1)$. Also, S–duality becomes the symmetry of the three–dimensional *action*. Moreover, both these groups turn out to be unified in a larger "U–duality" group $SO(2,p+2)$. This can be easily checked by computing the Kähler metric of the resulting target manifold. To find the σ–model representation one has to introduce a NUT potential χ via

$$d\chi = u^n dv^n - v^n du^n - f^2 * da, \tag{47}$$

and to derive the set of equations for χ, u^n in addition to the equations for f, κ, ϕ, v^n. The full set of equations will be that of the three–dimensional gravity coupled σ–model possessing the $4+2p$ dimensional target space $SO(2,2+p)/\left(SO(2)\times SO(p+2)\right)$. One can parametrize the target manifold by complex coordinates z^α, $\alpha = 0,1,\ldots,p+1$ which have the following meaning. The components $\alpha = n = 1,\ldots,p$ are complex potentials for vector fields

$$z^n = u^n - zv^n \equiv \Phi^n, \quad n = 1,\ldots,p, \tag{48}$$

the $\alpha = p+1$ component is the complex axidilaton field itself, $z^{p+1} = z$, and

$$z^0 = \epsilon - v^n\Phi^n \equiv E, \tag{49}$$

is the $N = 4$ analog of the Ernst potential.

Somewhat surprisingly, the Kähler potential of dilaton gravity remains untouched by the electric and magnetic potentials:

$$K = -\ln V, \quad V = \mathrm{Im}E\,\mathrm{Im}z + \left(\mathrm{Im}\Phi^n\right)^2 = f\mathrm{e}^{-2\phi}. \tag{50}$$

At the same time, being expressed through complex coordinates, Kähler potential has non–trivial dependence on all of them, so that the metric of the target space is non–degenerate.

Since the Kähler metric is given by mixed derivatives over holomorphic and antiholomorphic coordinates, a multiplication of V by an arbitrary holomorphic function and its complex conjugate (to preserve reality of V) does not change the metric. Thus a transformation

$$V(z,\bar{z}) \to f(z)\bar{f}(\bar{z})V(z,\bar{z}) \tag{51}$$

is the target space isometry. The Ernst–axidilaton duality (with Φ^n unchanged) belongs trivially to this class. Another useful discrete symmetry corresponds to

$$f(z) = \left(Ez+\Phi^2\right)^{-1}, \quad \Phi^2 \equiv \Phi^{n2}, \tag{52}$$

and consists in the following:

$$E \to \frac{z}{Ez+\Phi^2}, \quad z \to \frac{E}{Ez+\Phi^2}, \quad \Phi \to \frac{\Phi}{Ez+\Phi^2}. \tag{53}$$

Three–dimensional transformations $SO(2,2+p)$ of $N = 4$ supergravity can be listed in the following way. The most obvious symmetries include $p(p-1)/2$ $SO(p)$ rotations acting only on vector fields, $\Phi \to \Omega\Phi$, where $\Omega^T\Omega = I_p$, as well as $2p+1$ gauge transformations

$$\text{gravitational}: \qquad E \to E + g, \qquad \Phi, z \,\text{unchanged}, \tag{54}$$

$$\text{magnetic}: \qquad \Phi \to \Phi + m, \qquad E, z \,\text{unchanged}, \tag{55}$$

$$\text{electric}: \qquad \Phi \to \Phi + ez, \quad E \to E - 2e\Phi - e^2 z, \qquad z \,\text{unchanged}, \tag{56}$$

and scale

$$E \to \mathrm{e}^{2s} E, \quad \Phi \to \mathrm{e}^{s}\Phi, \quad z \,\text{unchanged}. \tag{57}$$

Here g, s, m, e are real scalar and vector group parameters. The remaining elements of the symmetry group include $2p+1$ Harrison–Ehlers transformations, which can be obtained by applying the above discrete maps. Namely, applying it to the electric gauge, one gets an electric Harrison transformation (the corresponding set of parameters h_e). Similarly, one obtains a magnetic Harrison (h_m) and Ehlers (c_E) transformations. The full group is closed by the $SL(2,R)$ S–duality expressed in terms of the target space variables. This three–parametric set can be obtained by applying discrete duality to gravitational gauge, scale and Ehlers transformation.

In the particular case $p = 1$, due to local isomorphism $SO(2,3) \sim Sp(4,R)$, there exists a simple matrix generalization of the Ernst potential . Let us form the (2×2) symmetric complex matrix collecting Kähler coordinates in the following way

$$\mathscr{E} = \begin{pmatrix} E & \Phi \\ \Phi & -z. \end{pmatrix} \tag{58}$$

One can easily check that the target space metric is reproduced via

$$dl^2 = -2\mathrm{Tr}\left\{ d\mathscr{E}\left(\bar{\mathscr{E}} - \mathscr{E}\right)^{-1} d\bar{\mathscr{E}}\left(\bar{\mathscr{E}} - \mathscr{E}\right)^{-1}\right\}, \tag{59}$$

which is a direct matrix analog of the vacuum gravity. Three–dimensional Einstein equations read:

$$\mathscr{R}_{ij} = -2\mathrm{Tr}\left\{\left(\bar{\mathscr{E}} - \mathscr{E}\right)^{-1}\left(\partial_{(i}\mathscr{E}\right)\left(\bar{\mathscr{E}} - \mathscr{E}\right)^{-1}\partial_{j)}\bar{\mathscr{E}}\right\}. \tag{60}$$

This is suggestive to express transformations as matrix-valued SL(2,R). The gauge transformation then is

$$\mathscr{E} \to \mathscr{E} + B, \tag{61}$$

where B is the real matrix of parameters

$$B = \begin{pmatrix} g & m \\ m & b \end{pmatrix}. \tag{62}$$

This matrix–valued transformation joins a gravitational gauge (g), magnetic gauge (m) and an axion shift (b) belonging to S–duality.

The scale transformation now is split into a symmetry preserving matrix relation:

$$\mathscr{E} \to A^T \mathscr{E} A. \tag{63}$$

Apart from the genuine $SL(2,R)$ scale (a), it includes gravitational scale (s), electric gauge (e) and electric Harrison (h_e) transformations:

$$A = \begin{pmatrix} \mathrm{e}^{s} & h_e \\ -e & a. \end{pmatrix} \tag{64}$$

The last subgroup is the linear shift of an inverted matrix

$$\mathscr{E}^{-1} \to \mathscr{E}^{-1} + C, \tag{65}$$

where C is a real symmetric matrix of parameters

$$C = \begin{pmatrix} c_E & h_m \\ h_m & c, \end{pmatrix} \tag{66}$$

combining c–transformation of S–duality with magnetic Harrison (h_m) and Ehlers (c_E) transformations. For pure dilaton–axion gravity without vector fields the matrices B, A, C become diagonal and correspond to the product of two $SL(2,R)$. Dilaton–axion gravity with one vector field generates $Sp(4,R)$ symmetry to make contact with the $Sp(4,R)$ group, one has merely to decompose the matrix Ernst potential into two symmetric real matrices

$$\mathscr{E} = \mathscr{Q} + i\mathscr{P}, \tag{67}$$

and then construct a 4×4 real matrix

$$\mathscr{M} = \begin{pmatrix} \mathscr{P}^{-1} & \mathscr{P}^{-1}\mathscr{Q} \\ \mathscr{Q}\mathscr{P}^{-1} & \mathscr{P} + \mathscr{Q}\mathscr{P}^{-1}\mathscr{Q} \end{pmatrix}. \tag{68}$$

This is a symmetric symplectic matrix satisfying

$$\mathscr{M}^T J \mathscr{M} = J, \quad J = \begin{pmatrix} O & I_2 \\ -I_2 & O. \end{pmatrix} \tag{69}$$

8. Five-dimensional minimal supergravity

This theory has structure similar to 11-dimensional supergravity: the action contains metric, and the two-form, with Chern-Simons term

$$S_{MSG5} = \int \left(\left[R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] \sqrt{g_5} - \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda \right) d^5x\,, \tag{70}$$

with $F = dA$. We compactify on a two-torus using

$$\begin{aligned} ds_5^2 &= \lambda_{ab}(dz^a + a_i^a dx^i)(dz^b + a_j^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j\,, && (71)\\ A_{(5)\mu} dx^\mu &= \sqrt{3}(\psi_a dz^a + A_i dx^i)\,, && (72) \end{aligned}$$

where $a, b = 0, 1$ and $\tau \equiv |\det\lambda|$. The $\nu = i$ components of the Maxwell-Chern-Simons equations allow to dualize the vector magnetic potential A_i to a scalar magnetic potential μ defined by

$$F^{ij} = a^{aj}\partial^i \psi_a - a^{ai}\partial^j \psi_a + \frac{1}{\tau\sqrt{h}} \epsilon^{ijk} \eta_k\,, \qquad \eta_k = \partial_k \mu + \epsilon^{ab} \psi_a \partial_k \psi_b\,. \tag{73}$$

The $\mu = i$, $\nu = a$ components of the Einstein equations are integrated by

$$\lambda_{ab} G^{bij} = \frac{1}{\tau\sqrt{h}} \epsilon^{ijk} V_{ak}\,, \qquad V_{ak} = \partial_k \omega_a - \psi_a \left(3\partial_k \mu + \epsilon^{bc} \psi_b \partial_k \psi_c \right)\,, \tag{74}$$

where $G^b = da^b$, and ω_a is the twist two-potential. The $D = 3$ sigma model is then obtained with eight target space coordinates $\Phi^A = \{\lambda_{ab}, \omega_a, \psi_a, \mu\}$ and metric

$$dl^2 = \frac{1}{2}\mathrm{Tr}(\lambda^{-1}d\lambda\lambda^{-1}d\lambda) + \frac{1}{2}\tau^{-2}d\tau^2 - \tau^{-1}V^T\lambda^{-1}V + 3\left(d\psi^T\lambda^{-1}d\psi - \tau^{-1}\eta^2\right). \qquad (75)$$

This space has 14 Killing vectors. Nine manifest gauge transformations , grouped according to their transformations under $GL(2R)$ (the group of linear transformations in the (z^1, z^2) plane) into the quadruplet

$$M_a{}^b = 2\lambda_{ac}\frac{\partial}{\partial\lambda_{cb}} + \omega_a\frac{\partial}{\partial\omega_b} + \delta^b_a\omega_c\frac{\partial}{\partial\omega_c} + \psi_a\frac{\partial}{\partial\psi_b} + \delta^b_a\mu\frac{\partial}{\partial\mu} \qquad (76)$$

(the generators of the $gl(2,R)$ subalgebra), the doublet and the singlet associated with the the three cyclic ‘magnetic’ coordinates:

$$N^a = \frac{\partial}{\partial\omega_a}, \quad Q = \frac{\partial}{\partial\mu}, \qquad (77)$$

and the doublet generating infinitesimal gauge transformations of the ψ_a

$$R^a = \frac{\partial}{\partial\psi_a} + 3\mu\frac{\partial}{\partial\omega_a} - \epsilon^{ab}\psi_b\left(\frac{\partial}{\partial\mu} + \psi_c\frac{\partial}{\partial\omega_c}\right). \qquad (78)$$

The five remaining, non trivial infinitesimal isometries L_a, P_a and T closing the Lie algebra $g_{2(2)}$ are more complicated. The L_a, $M_a{}^b$ and N^a generate the vacuum subalgebra $sl(3,R)$. Assuming a spacelike two-torus, the target space (75) is identified as the coset space $G_{2(2)}/(SU(2)\times SU(2))$.

The 7×7 symmetric matrix representative of the coset obtained by exponentiation of the Borel subalgebra

The first step is to construct real matrix representatives of $O(4,3)$, beginning with the subalgebra $O(3,3) \sim sl(4,R)$. Rather than using the Maison parametrisation of $sl(4,R)$ in terms of 4×4 matrices (which presumably would lead to a representation of $O(4,3)$ in terms of 8×8 matrices), we use the representation of $O(3,3)$ in terms of 6×6 matrices, decomposed in 3×3 blocks according to

$$M_a{}^b = \begin{pmatrix} m_a{}^b & 0 \\ 0 & -\tilde{m}^b_a \end{pmatrix}, \quad N^a = \begin{pmatrix} 0 & n^a \\ 0 & 0 \end{pmatrix}, \quad L_a = \begin{pmatrix} 0 & 0 \\ -n^{aT} & 0 \end{pmatrix}, \qquad (79)$$

where $\tilde{}$ denotes the anti-transposition, i.e. transposition relative to the anti- (or minor) diagonal, and

$$(m_a{}^b)^\alpha{}_\beta = \delta^\alpha_a\delta^b_\beta - \delta^b_a\delta^\alpha_\beta, \qquad (80)$$

$$n^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad n^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$(\alpha, \beta = 1,2,3)$. These matrices satisfy the commutation relations.

The 7×7 matrix generators of $O(4,3)$ contain the preceding, promoted to 7×7 matrices by the addition of a central 3-row and a central 3-column, in block form

$$M_a{}^b = \begin{pmatrix} m_a{}^b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tilde{m}_a^b \end{pmatrix}, N^a = \begin{pmatrix} 0 & 0 & n^a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -n^{aT} & 0 & 0 \end{pmatrix}, \tag{81}$$

together with

$$R_a = \sqrt{2}\begin{pmatrix} 0 & r_a & 0 \\ 0 & 0 & -\tilde{r}_a \\ 0 & 0 & 0 \end{pmatrix}, \quad P^a = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ r_a^T & 0 & 0 \\ 0 & -\tilde{r}_a^T & 0 \end{pmatrix}, \tag{82}$$

where r_a is the column matrix of elements

$$r_a^\alpha = \delta_a^\alpha. \tag{83}$$

Using

$$r_a\tilde{r}_b - r_b\tilde{r}_a = \epsilon_{abc}n^c, \tag{84}$$

these can be checked to satisfy the remaining commutation relations of $O(4,3)$.

The 7×7 coset matrix representative is

$$\mathcal{M} = \mathcal{V}^T\mathcal{M}_0\mathcal{V}, \tag{85}$$

with

$$\mathcal{M}_0 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \tilde{\mu}^{-1} \end{pmatrix}, \quad \mu = \tau^{-1}\lambda, \tag{86}$$

and

$$\mathcal{V} = e^{\chi^a R_a}e^{\omega_a N^a} = \begin{pmatrix} 1 & \sqrt{2}\chi & \gamma \\ 0 & 1 & -\sqrt{2}\tilde{\chi} \\ 0 & 0 & 1 \end{pmatrix}, \tag{87}$$

where

$$\chi = \begin{pmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{pmatrix}, \quad \gamma = \hat{\omega} - \chi\tilde{\chi}, \quad \hat{\omega} = \begin{pmatrix} -\omega_2 & \omega_3 & 0 \\ \omega_1 & 0 & -\omega_3 \\ 0 & -\omega_1 & \omega_2 \end{pmatrix}. \tag{88}$$

The resulting coset representative

$$\mathcal{M} = \begin{pmatrix} \mu & \sqrt{2}\mu\chi & \mu\gamma \\ \sqrt{2}\chi^T\mu & -1+2\chi^T\mu\chi & \sqrt{2}(\chi^T\mu\gamma+\tilde{\chi}) \\ \gamma^T\mu & \sqrt{2}(\gamma^T\mu\chi+\tilde{\chi}^T) & \gamma^T\mu\gamma - 2\tilde{\chi}^T\tilde{\chi}+\tilde{\mu}^{-1} \end{pmatrix} \tag{89}$$

is related to its inverse by

$$\mathcal{M}^{-1} = \tilde{\mathcal{M}} \tag{90}$$

(use $\tilde{\mathcal{V}}(\omega,\chi) = \mathcal{V}(-\omega,-\chi)$).

9. Six-dimensional supergravity with self-dual three-form

Our last example is six-dimensional minimal supergravity containing the metric and self-dual three-form field

$$S_{MSG6} = \int \left(\hat{R} - \frac{1}{12} \hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} \hat{G}^{\hat{\mu}\hat{\nu}\hat{\lambda}} \right) \sqrt{-\hat{g}} d^6 x, \tag{91}$$

where $\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} \equiv 3\hat{C}_{[\hat{\mu}\hat{\nu},\hat{\lambda}]}$, with subsidiary condition

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} = \frac{1}{6} \sqrt{-\hat{g}} \epsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}} \hat{\rho}\hat{\sigma}\hat{\tau} \hat{G}^{\hat{\rho}\hat{\sigma}\hat{\tau}}, \tag{92}$$

which has to be imposed after variation of the action The action (91) is a lowest-dimensional member of the even-dimensional sequence of actions containing self-dual form fields, the largest representative of which is the IIB ten-dimensional supergravity.

Somewhat unexpectedly, this action, being compactified on a circle, turns out to be non-locally dual to the truncated five-dimensional heterotic string effective action which belongs to another sequence of the string actions.

Simple representation of the coset $O(4,3)/(O(4) \times O(3)$ may be achieved by performing direct compactification of the six-dimensional theory on T^3. We start with the Lagrangian (91), and assume 3 Killing vectors ∂_a ($a = 1,2,3$). The six-dimensional metric and 3-form may be parameterized by

$$\begin{aligned} ds_6^2 &= \lambda_{ab}(dz^a + a_i^a dx^i)(dz^b + a_j^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j, \\ \hat{G}_{abc} &= 0, \quad \hat{G}_{abi} = \hat{B}_{ab,i}, \end{aligned} \tag{93}$$

($\tau \equiv -\det\lambda,\ i,j = 4,5,6$) and the 10 remaining components of $\hat{G}$ related to these by self-duality. Put

$$\hat{B}_{ab} \equiv \epsilon_{abc} \chi^c. \tag{94}$$

Then,

$$\hat{G}_{abi} = \epsilon_{abc} \chi^c_{,i}, \quad \hat{G}^{aij} = -\frac{\tau}{\sqrt{h}} \epsilon^{ijk} \chi^a_{,k}. \tag{95}$$

The mixed Einstein equations

$$\begin{aligned} \hat{R}^i_a &\equiv \frac{\tau}{2\sqrt{h}} \partial_j [\tau \sqrt{h} \lambda_{ab} \mathscr{F}^{bij}] \\ &= \frac{1}{2} \hat{G}^{ibj} \hat{G}_{abj} = \frac{\tau}{2\sqrt{h}} \partial_j [\epsilon^{ijk} \epsilon_{abc} \chi^b_{,k} \chi^c] \end{aligned} \tag{96}$$

($\mathscr{F}^b \equiv da^b$)

are solved by

$$\lambda_{ab} \mathscr{F}^{bij} = \frac{1}{\tau\sqrt{h}} \epsilon^{ijk} V_{ak}, \quad V_{ak} \equiv \partial_k \omega_a + \epsilon_{abc} \chi^b_{,k} \chi^c. \tag{97}$$

The remaining Einstein equations then lead to the gravitating sigma model with target space metric

$$dl^2 = \frac{1}{2} \mathrm{Tr}(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{2} \tau^{-2} d\tau^2 - \tau^{-1} V^T \lambda^{-1} V - 2\tau^{-1} d\chi^T \lambda d\chi, \tag{98}$$

where

$$V \equiv d\omega - \chi \wedge d\chi \,. \tag{99}$$

The dimension of this target space is twelve: six components of the symmetric matrix λ_{ab} and two triplets ω_a, χ^a. In Appendix A we check that it admits 21 Killing vectors generating the Lie algebra $o(4,3)$. These include nine Killing vectors ${M_a}^b$ generating the algebra $gl(3,R)$ of linear transformations in the three-Killing vector space, six vectors N^a and L_a which together with the ${M_a}^b$ generate the isometry algebra $sl(4,R)$ for the target subspace corresponding to the six-dimensional vacuum sector, and six more vectors R_a and P^a which complete the algebra $o(4,3)$.

The fifteen Killing vectors ${M_a}^b$, N^a and R_a generate generalized gauge transformations, with the N^a generating translations of the twists ω_a and the R_a generating gauge transformations of the χ^a.

The 7×7 coset matrix representative is then constructed in a standard fashion as

$$\mathcal{M} = \mathcal{N}^T \eta \mathcal{N} \,, \tag{100}$$

where $\mathcal{N}$ is obtained by exponentiating a suitable Borel subalgebra of $0(4,3)$, and η is a suitably chosen constant matrix. The resulting coset representative is, in block form,

$$\mathcal{M} = \begin{pmatrix} \mu & \sqrt{2}\mu\chi & \mu\gamma \\ \sqrt{2}\chi^T\mu & -1+2\chi^T\mu\chi & \sqrt{2}(\chi^T\mu\gamma+\tilde{\chi}) \\ \gamma^T\mu & \sqrt{2}(\gamma^T\mu\chi+\tilde{\chi}^T) & \gamma^T\mu\gamma - 2\tilde{\chi}^T\tilde{\chi}+\tilde{\mu}^{-1} \end{pmatrix} \tag{101}$$

where˜denotes the anti-transposition, i.e. transposition relative to the anti- (or minor) diagonal, and

$$\mu = \tau^{-1}\lambda = \tau^{-1}\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix},$$

$$\chi = \begin{pmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{pmatrix}, \; \tilde{\chi} = (\chi_3, \chi_2, \chi_1), \; \gamma = \hat{\omega} - \chi\tilde{\chi},$$

$$\hat{\omega} = \begin{pmatrix} -\omega_2 & \omega_3 & 0 \\ \omega_1 & 0 & -\omega_3 \\ 0 & -\omega_1 & \omega_2 \end{pmatrix}. \tag{102}$$

One can check that the target space metric (98) can be expressed as usual

$$dl^2 = \frac{1}{4}\mathrm{Tr}(\mathcal{M}^{-1}d\mathcal{M}\mathcal{M}^{-1}d\mathcal{M})\,. \tag{103}$$

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